

Analysis Of Variance



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Introduction

ANOVA is the technique where the total variance present in the data set is spilt up into non- negative components where each component is due to one factor or cause of variation.

Factors of variation

Assignable

Can be many

Non-assignable

Error or Random
variation

Utility

- ANOVA is used to test hypotheses about differences between two or more means.
- The t-test can only be used to test differences between two means.
- When there are more than two means, it is possible to compare each mean with each other mean using t-tests.
- However, conducting multiple t-tests can lead to severe inflation of the Type_I error rate.
- ANOVA can be used to test differences among several means for significance without increasing the Type I error rate.

The ANOVA Procedure:

This is the ten step procedure for analysis of variance:

1.Description of data

2.Assumption: Along with the assumptions, we represent the model for each design we discuss.

3. Hypothesis

4.Test statistic

5.Distribution of test statistic

6.Decision rule

7.Calculation of test statistic: The results of the arithmetic calculations will be summarized in a table called the analysis of variance (ANOVA) table. The entries in the table make it easy to evaluate the results of the analysis.

8.Statistical decision

9.Conclusion

10.Determination of p value

ONE-WAY ANOVA- Completely Randomized Design (CRD)

One-way ANOVA:

- It is the simplest type of ANOVA, in which only one source of variation, or factor, is investigated.
- It is an extension to three or more samples of the t test procedure for use with two independent samples
- In another way t test for use with two independent samples is a special case of one-way analysis of variance.

- *Experimental design used for one-way ANOVA is called Completely randomised design.*
 - *This test the effect of equality of several treatments of one assignable cause of variation.*
 - *Based on two principles- Replication and randomization.*

Advantages:

- *Very simple:*
- *Reduce the experimental error to a great extent.*
- *We can reduce or increase some treatments.*
- *Suitable for laboratory experiment.*

Disadvantages: Design is not suitable if the experimental units are not homogeneous.

- *Design is not so much efficient and sensitive as compared to others.*
- *Local control is completely neglected.*
- *Not suitable for field experiment.*

Hypothesis Testing Steps:

1. Description of data: The measurements(or observation) resulting from a completely randomized experimental design, along with the means and totals.

Available
Subjects

01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16



Random
numbers

16 09 06 15 14 11 02 04 10 07 05 13 03 12 01 08

16 09 06 15 14 11 02 04 10 07 05 13 03 12 01 08

Table of Sample Values for the CRD

	Treatment				
	1	2	3 ...	K	
	x_{11}	x_{12}	x_{13} ...	x_{1k}	
	x_{21}	x_{22}	x_{23}	x_{2k}	
	.	.	.	.	
	$x_{n_1 1}$	$x_{n_2 2}$	$x_{n_3 3}$	$x_{n_k k}$	
Total	$T_{.1}$	$T_{.2}$	$T_{.3}$	$T_{.k}$	$T_{..}$
Mean	$\bar{x}_{.1}$	$\bar{x}_{.2}$	$\bar{x}_{.3}$	$\bar{x}_{.k}$	$\bar{x}_{..}$

x_{ij} = the i^{th} observation resulting from the j^{th} treatment (there are a total of k treatment)

$T_j = \sum x_{ij}$ = total of the j^{th} treatment

$\bar{x}_{.j} = \frac{T_j}{n_j}$ = mean of the j^{th} treatment

$T_{..} = \sum T_j = \sum \sum x_{ij}$ = total of all observations

$\bar{x}_{..} = \frac{T_{..}}{N}$, $N = \sum n_j$

2. Assumption:

The Model

- The one-way analysis of variance may be written as follows:

$$x_{ij} = \mu + \tau_j + e_{ij}; i=1,2\dots n_j, j= 1,2\dots k$$

The terms in this model are defined as follows:

1. μ represents the mean of all the k population means and is called the grand mean.
2. τ_j represents the difference between the mean of the j^{th} population and the grand mean and is called the treatment effect.
3. e_{ij} represents the amount by which an individual measurement differs from the mean of the population to which it belongs and is called the error term.

Assumptions of the Model

- The k sets of observed data constitute k independent random samples from the respective populations.
- Each of the populations from which the samples come is normally distributed with mean μ_j and variance σ_j^2 .
- Each of the populations has the same variance. That is $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2 = \sigma^2$, the common variance.
- The τ_j are unknown constants and $\sum \tau_j = 0$, since the sum of all deviations of the μ_j from their mean, μ , is zero.
- The e_{ij} have a mean of 0, since the mean of x_{ij} is μ_j
- The e_{ij} have a variance equal to the variance of the x_{ij} , since the e_{ij} and x_{ij} differ only by a constant.
- The e_{ij} are normally (and independently) distributed.

3. Hypothesis:

We test the null hypothesis that all population or treatment means are equal against the alternative that the members of at least one pair are not equal. We may state the hypothesis as follows

$$H_0: \mu_1 = \mu_2 = \dots = \mu_k$$

$$H_A: \text{not all } \mu_j \text{ are equal}$$

If the population means are equal, each treatment effect is equal to zero, so that alternatively, the hypothesis may be stated as

$$H_0: \tau_j = 0, j=1,2,\dots,k$$

$$H_A: \text{not all } \tau_j = 0$$

4. Test statistic:

Table: Analysis of Variance Table for the Completely Randomized Design

Source of variation	Sum of square	d.f	Mean square	Variance ratio
Among sample	$SSA = \sum_{j=1}^k n_j (\bar{x}_{.j} - \bar{x}_{..})^2$	k-1	MSA=SSA/(k-1) MS due to Treatment	V.R=MSA/MSW
Within samples	$SSW = \sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x}_{.j})^2$	N-k	MSW=SSW/(N-k) MS due to error	
Total	$SST = \sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x}_{..})^2$	N-1		

The Total Sum of squares(SST): It is the sum of the squares of the deviations of individual observations taken together.

The Within Groups of Sum of Squares:

The first step in the computation call for performing some calculations within each group. These calculation involve computing within each group the sum of squared deviations of the individual observations from their mean. When these calculations have been performed within each group, we obtain the sum of the individual group results.

The Among Groups Sum of Squares:

To obtain the second component of the total sum of square, we compute for each group the squared deviation of the group mean from the grand mean and multiply the result by the size of the group. Finally we add these results over all groups. Total sum of square is equal to the sum of the among and the within sum of square.

$$SST=SSA+SSW$$

The First Estimate of σ^2 :

Within any sample

$$\frac{\sum_{j=1}^n (x_{ij} - \bar{x}_{.j})^2}{n_j - 1}$$

Provides an unbiased estimate of the true variance of the population from which the sample came. Under the assumption that the population variances are all equal, we may pool the k estimate to obtain

$$\frac{\sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x}_{.j})^2}{\sum_{j=1}^k (n_j - 1)}$$

The Second Estimate of σ^2 :

The second estimate of σ^2 may be obtained from the familiar formula for the variance of sample means, $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$. If we solve this equation for σ^2 , the variance of the population from which the samples were drawn, we have

$$\sigma^2 = n \sigma_{\bar{x}}^2$$

An unbiased estimate of $\sigma_{\bar{x}}^2$, computed from sample data, is provided by

$$\frac{\sum_{j=1}^k (\bar{x}_{.j} - \bar{x}_{..})^2}{k - 1}$$

If we substitute this quantity into equation we obtain the desired estimate of σ^2

$$\frac{n \sum_{j=1}^k (\bar{x}_{.j} - \bar{x}_{..})^2}{k - 1}$$

When the sample sizes are not all equal, an estimate of σ^2 based on the variability among sample means is provided by

$$\frac{\sum_{j=1}^k n_j (\bar{x}_{.j} - \bar{x}_{..})^2}{k-1}$$

The Variance Ratio:

What we need to do now is to compare these two estimates of σ^2 , and we do this by computing the following variance ratio, which is the desired test statistic:

$$\text{V.R} = \frac{\text{Among groups mean square}}{\text{Within groups mean square}}$$

6. Distribution of Test statistic:

F distribution we use in a given situation depends on the number of degrees of freedom associated with the sample variance in the numerator and the number of degrees of freedom associated with the sample variance in the denominator.

we compute V.R. in situations of this type by placing the among groups mean square in the numerator and the within groups mean square in the denominator , so that the numerator degrees of freedom is equal to the number of groups minus 1, (k-1), and the denominator degrees of freedom value is equal to

$$\sum_{j=1}^k (n_j - 1) = \sum_{j=1}^k n_j - k = N - k$$

7. Significance Level:

Once the appropriate F distribution has been determined, the size of the observed V.R. that will cause rejection of the hypothesis of equal population variances depends on the significance level chosen. The significance level chosen determines the critical value of F, the value that separates the nonrejection region from the rejection region.

8. Statistical decision:

To reach a decision we must compare our computed V.R. with the critical value of F, which we obtain by entering Table G with $k-1$ numerator degrees of freedom and $N-k$ denominator degrees of freedom .

If the computed V.R. is equal to or greater than the critical value of F, we reject the null hypothesis. If the computed value of V.R. is smaller than the critical value of F, we do not reject the null hypothesis.

9. Conclusion:

When we reject H_0 we conclude that not all population means are equal. When we fail to reject H_0 , we conclude that the population means may be equal.

10. Determination of p value

Example:1

The aim of a study by Makynen et al.(A-1) was to investigate whether increased dietary calcium as a nonpharmacological treatment of elevated blood pressure could beneficially influence endothelial function in experimental mineralocorticoid-NaCl hypertension. The researchers divided seven week-old male Wistar –Kyoto rats (WKY) into four groups with equal mean systolic blood pressure: untreated rats on normal(WKY) and high-calcium(WKY-Ca) diets, and deoxycorticosterone-NaCl-treated rats on normal(DOC) and high-calcium diets(DOC-Ca). We wish to know if the four conditions have different effects on the mean weights of male rats.

Condition					
	Doc	WKY	DOC-Ca	WKY-Ca	
	336	328	304	342	
	346	315	292	284	
	269	343	299	334	
	346	368	293	348	
	323	353	277	315	
	309	374	303	313	
	322	356	303	301	
	316	339	320	354	
	300	343	324	346	
	309	343	340	319	
	276	334	299	289	
	306	333	279	322	
	310	313	305	308	
	302	333	290	325	
	269	372	300		
	311		312		
Total	4950	5147	4840	4500	19437
Mean	309.38	343.13	302.50	321.43	318.64

Assumption: We assume that the four sets of data constitute independent simple random samples from four populations that are similar expect for the condition studied. We assume that the four populations of measurements are normally distributed with equal variances.

Hypothesis: $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ (On the average the four conditions elicit the same response)

H_A : Not all μ 's are equal

Test statistic: The test statistic is $V.R = MSA/MSW$.

Source	SS	d.f	MS	V.R
Among samples	14649.1514	3	4883.0503	11.99
Within samples	23210.9023	57	407.2088	
Total	37860.0547	60		

Distribution of test statistic: If H_0 is true and the assumptions are met, V.R follows the F distribution with $4-1=3$ numerator degrees of freedom and $61-4=57$ denominator degrees of freedom.

Decision rule: Suppose let $\alpha=0.05$. The critical value of F from Table G is 3.34. The decision rule, then, is reject H_0 if the computed V.R is equal to or greater than 3.34.

Calculation of test statistic:

$$SST=37860.0547$$

$$SSA=14649.1514$$

$$SSW=37860.0547-14649.1514=23210.9023$$

Statistical decision: Since our computed V.R of 11.99 is greater than the critical F of 3.34, we reject H_0 .

Conclusion: Since we reject H_0 , the four treatments do not all have the same average effect.

p value: Since $11.99 > 4.77$, $p < 0.005$ for this test.

Testing for Significant Differences Between Individual Pairs of Means: Turkey's HSD Test:

$$HSD = q_{\alpha, k, N-k} \sqrt{\frac{MSE}{n}}$$

Turkeys Test for unequal sample sizes (Spjotvoll and Stolins)

$$HSD^* = q_{\alpha, k, N-k} \sqrt{\frac{MSE}{n_j^*}}$$

n_j^* = smallest of the two sample sizes that are compared.

Absolute value of the difference between the two corresponding sample means if it exceeds HSD^* is declared to be significant

Example:2

Let us illustrate the use of the HSD test with the data from the Example-1.

Solution: The first step is to prepare a table of all possible (ordered) differences between means. This is displayed in the following table:

	DOC-Ca	DOC	WKY-Ca	WKY
DOC-Ca (DC)	-	6.87	18.93	40.63
DOC (D)		-	12.06	33.76
WKY-Ca (WC)			-	21.70
WKY(W)				-

Suppose we let $\alpha = 0.05$. Entering table H with $\alpha = 0.05$, $k=4$, and $N-k=57$, we find that $q = 3.75$. $MSE=407.2088$.

The hypothesis that can be tested, the value of HSD*, and the statistical decision for each test are shown in the following Table.

Table: Multiple Comparison Tests Using data of Example 1 and HSD*

Hypothesis	HSD*	Statistical Decision
$H_0: \mu_{DC} = \mu_D$	$HSD^* = 3.75 \sqrt{\frac{407.2088}{16}} = 18.92$	Do not reject H_0 since $6.87 < 18.92$
$H_0: \mu_{DC} = \mu_{WC}$	$HSD^* = 3.75 \sqrt{\frac{407.2088}{14}} = 20.22$	Do not reject H_0 since $18.93 < 20.22$
$H_0: \mu_{DC} = \mu_W$	$HSD^* = 3.75 \sqrt{\frac{407.2088}{15}} = 19.54$	reject H_0 since $40.63 > 19.54$
$H_0: \mu_D = \mu_{WC}$	$HSD^* = 3.75 \sqrt{\frac{407.2088}{14}} = 20.22$	Do not reject H_0 since $12.06 < 20.22$
$H_0: \mu_D = \mu_W$	$HSD^* = 3.75 \sqrt{\frac{407.2088}{15}} = 19.54$	reject H_0 since $33.76 > 19.54$
$H_0: \mu_{WC} = \mu_W$	$HSD^* = 3.75 \sqrt{\frac{407.2088}{14}} = 20.22$	Do not reject H_0 since $21.7 < 20.22$

Randomized Block design(RBD)

- In RBD three principle of design is used i.e. replication, randomization and local control and randomization is restrict to only one direction.

Advantages

- Simplest method to test the treatment effects as well as block effects.
- Statistical analysis also simple because it is based on two-way classification.
- More efficient than CRD.
- Trend effect is reduced.
- Suitable for field experiment as well as lab. Experiment.

Disadvantages

- If the treatments are more than the design is not suitable.

Table of Sample Values for the Randomized Complete Block Design

	Treatments					
Blocks	1	2	3	k	Total	Mean
1	x_{11}	x_{12}	x_{13}	x_{1k}	$T_{1.}$	$\bar{x}_{1.}$
2	x_{21}	x_{22}	x_{23}	x_{2k}	$T_{2.}$	$\bar{x}_{2.}$
.						
.						
n	x_{n1}	x_{n2}	x_{n3}	x_{nk}	$T_{n.}$	$\bar{x}_{n.}$
Total	$T_{.1}$	$T_{.2}$	$T_{.3}$	$T_{.k}$	$T_{..}$	
Mean	$\bar{x}_{.1}$	$\bar{x}_{.2}$	$\bar{x}_{.3}$	$\bar{x}_{.k}$		$\bar{x}_{..}$

The Model for two-way ANOVA is

$$x_{ij} = \mu + \beta_i + \tau_j + e_{ij}, i = 1, 2, \dots, n; j = 1, 2, \dots, k$$

In this model x_{ij} is a typical value from the overall population.

μ is an unknown constant.

β_i represents a block effect reflecting the fact that the experimental unit fell in the i th block.

τ_j represents a treatment effect, reflecting the fact that the experimental unit received the j th treatment

e_{ij} is a residual component representing all sources of variation other than treatments and blocks.

Assumptions of the Model:

- (a) Each x_{ij} that is observed constitute a random independent sample of size 1 from one of the kn populations represented.
- (b) each of these kn populations is normally distributed with mean μ_{ij} and variance σ^2 . e_{ij} are independently and normally distributed with mean 0 and variance σ^2 .
- (c) The block and treatment effects are additive.

Example:3

A physical therapist wished to compare three methods for teaching patients to use a certain prosthetic device. He felt that the rate of learning would be different for patients of different ages and wished to design an experiment in which the influence of age could be taken into account.

Solution:

Assumption: We assume that each of the 15 observations constitutes a simple random of size 1 from one of the 15 populations defined by a block-treatment combination.

Hypothesis:

$$H_0: \tau_j = 0, j=1,2,3$$

$$H_A: \text{not all } \tau_j = 0$$

$$\text{Let } \alpha = 0.05$$

Test statistic: The test statistic is $MSTr/MSE$

Distribution of test statistic: When H_0 is true and the assumptions are met, V.R follows an F distribution with 2 and 8 degrees of freedom

Table: Time(in days)required to learn the use of a certain Prosthetic device

Teaching methods					
Age group	A	B	C	Total	Mean
Under 20	7	9	10	26	8.67
20 to 29	8	9	10	27	9.00
30 to 39	9	9	12	30	10.00
40 to 49	10	9	12	31	10.33
50 and above	11	12	14	37	12.33
Total	45	48	58	151	
Mean	9.0	9.6	11.6		10.7

Decision rule: reject the null hypothesis if the computed V.R is equal to or greater than the critical F, which we find in table G to be 4.46.

Calculation of test statistic:

$$SST = (7-10.7)^2 + (8-10.07)^2 + \dots + (14-10.07)^2 = 46.9335$$

$$SSBI = 3[(8.67-10.07)^2 + (9.00-10.07)^2 + \dots + (12.33-10.07)^2] \\ = 24.855$$

$$SSTr = 5[(9-10.07)^2 + (9.6-10.07)^2 + (11.6-10.07)^2] = 18.5335$$

$$SSE = 46.9335 - 24.855 - 18.5335 = 3.545$$

The degrees of freedom are total = $(3)(5)-1=14$, blocks = $5-1=4$, treatments = $3-1=2$, and residual = $(5-1)(3-1)=8$. The results of the calculations may be displayed in an ANOVA Table.

Source	Ss	d.f	MS	V.R
Treatments	18.5335	2	9.26675	20.91
Blocks	24.855	4	6.21375	
Residual	3.545	8	0.443125	
Total	46.9335	14		

Statistical decision: Since our computed variance ratio, 20.91, is greater than 4.46, we reject the null hypothesis of no treatment effects on the assumption that such a large V.R reflects the fact that the two sample mean square are not estimating the same quantity. The only other explanation for this large V.R would be that the null hypothesis is really true, and we have just observed an unusual set of results.

Conclusion: We conclude that not all treatment effects are equal to zero, or equivalently, that not all treatment means are equal.

p value: For this test $p < 0.005$

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THANK YOU